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ON CERTAIN SINGULARITIES OF THE HESSIANS OF THE CUBIC AND THE QUARTIC.

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The Hessian of a curve of the nth degree,

$$u = \varphi(x, y, z) = 0,$$

is a curve of the $3(n-2)$th degree,

$$\begin{vmatrix} \frac{d^2u}{dx^2}, & \frac{d^2u}{dxdy}, & \frac{d^2u}{dxdz} \\ \frac{d^2u}{dydx}, & \frac{d^2u}{dy^2}, & \frac{d^2u}{dydz} \\ \frac{d^2u}{dzdx}, & \frac{d^2u}{dzdy}, & \frac{d^2u}{dz^2} \end{vmatrix} = 0.$$

It is easy to prove[*] that the Hessian passes through the points of inflexion of u, and also[†] that if a point be a multiple point of order k on u, it is a multiple point of order $3k - 4$ on the Hessian. The question may then be proposed: Given the proposition that a point is a singular point on u, what is the nature of its singularity on the Hessian? I shall attempt to answer this question for the cases of cubics and quartics. By placing $z = 1$ and taking the singular point as the origin, the discussion will be greatly simplified, as the equation will then lose one or more dimensions.

THE CUBIC.

The general equation of the cubic may be written

$$ax^3 + by^3 + cz^3 + 3a'x^2y + 3a''x^2z + 3b'y^2x + 3b''y^2z + 3c'z^2x + 3c''z^2y + 6mxyz = 0.$$

The equation of its Hessian is[‡]

$$\begin{aligned} &(ab'c' - am^2 + 2a'a''m - a''^2b' - a'^2c')\,x^3 \\ &+ (a'bc'' - bm^2 + 2b'b''m - a'b''^2 - b'^2c'')\,y^3 \\ &+ (a''b''c - cm^2 + 2c'c''m - a''c''^2 - b''c'^2)\,z^3 \\ &+ (abc' - 2ab''m + ab'c'' - a''^2b + a'm^2 - a'b'c' + 2a'a''b'' - a'^2c'')\,x^2y \\ &+ \left\{\begin{matrix} ab'c - 2ac''m + ab''c' - a'^2c + a''m^2 \\ - a''b'c' + 2a'a''c'' - a''^2b'' \end{matrix}\right\} x^2z \end{aligned}$$

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[*] Salmon's Higher Plane Curves, 3d Edition, Art. 74.

[†] Same, Art. 75.

[‡] Same, Art. 218.

$$\begin{aligned}
&+ (abc'' - 2a''bm + a'bc' - ab''^2 + b'm^2 - a'b'c'' + 2a''b'b'' - b'^2c')\, y^2x \\
&+ (a'bc - 2bc'm + a''bc'' - b'^2c + b''m^2 - a'b''c'' + 2b'b''c' - a''b''^2)\, y^2z \\
&+ (ab''c - 2a'cm + a''b'c - ac''^2 + c'm^2 - a''b''c' + 2a'c'c'' - b'c'^2)\, z^2x \\
&+ (a''bc - 2b'cm + a'b''c - bc'^2 + c''m^2 - a''b''c'' + 2b'c'c'' - a'c''^2)\, z^2y \\
&\qquad + \left\{ \begin{array}{c} abc - ab''c'' - a''bc' - a'b'c + 2m^3 - 2b'c'm \\ - 2a'c''m - 2a''b''m + 3a'b''c' + 3a''bc'' \end{array} \right\} xyz = 0.
\end{aligned}$$

If the origin is a double point on the cubic, all terms of a degree below the second vanish; that is

$$c = c' = c'' = 0.$$

The cubic is then

$$ax^3 + 3a'x^2y + 3b'xy^2 + by^3 + 3a''x^2 + 6mxy + 3b''y^2 = 0.$$

The equation to the tangents at the double point found by equating to zero the terms of the second degree is

$$a''x^2 + 2mxy + b''y^2 = 0.$$

The equation of the Hessian becomes

$$\begin{aligned}
&(2a'a''m - am^2 - a''^2b')\, x^3 \\
&+ (a'm^2 - a''^2b - 2ab''m + 2a'a''b'')\, x^2y \\
&+ (b'm^2 - ab''^2 - 2a''bm + 2a''b'b'')\, xy^2 \\
&+ (2b'b''m - bm^2 - a'b''^2)\, y^3 \\
&+ (m^2 - a''b'')(a''x^2 + 2mxy + b''y^2) = 0.
\end{aligned}$$

The origin is a double point on the Hessian, as has been already stated; but also the tangents at the double point on the Hessian coincide with the tangents at the double point on the cubic, for their equation is

$$a''x^2 + 2mxy + b''y^2 = 0.$$

The proof of this last proposition given by Salmon (Art. 75) seems to apply only to the case of real tangents.

The Hessian is, then, crunodal or acnodal according as the tangents at the double point on u are real or imaginary. We might infer that the Hessian of a cuspidal cubic is a cuspidal cubic, but such is not the case. In fact, it may be proved generally (Salmon, Art. 77) that a cusp on u is a triple point on the Hessian. Following our line of proof, the condition that the tangents at the double point on u may coincide is

$$m^2 - a''b'' = 0;$$

but reverting to the equation of the Hessian, we see that this quantity is a factor of all terms of the second degree, and therefore the equation contains no terms of lower degree than the third. The origin is then a triple point. But a cubic cannot have a triple point unless it break up into three right lines. Writing the equation of the Hessian in the form

$$(a''x^2 + 2mxy + b''y^2)\left[(2a'm - ab'' - a''b')\,x + (2b'm - a''b - a'b'')\,y\right] = 0,$$

we infer that the Hessian of a cuspidal cubic is three right lines, two of which coincide with the tangent at the cusp on u. For example, the Hessian of the cissoid

$$x\,(x^2 + y^2) = 2ry^2,$$

and also the semi-cubical parabola

$$ry^2 = x^3,$$

is

$$xy^2 = 0,$$

which represents three right lines, two coinciding with the axis of x and the other with the axis of y.

Suppose that

$$a'' = c = c' = 0.$$

The origin is a point of inflexion on u, and $y = 0$, the tangent at it. The Hessian is

$$\begin{aligned} &- am^2x^3 \\ &+ (ab'c'' - 2ab''m + a'm^2 - a'^2c'')\,x^2y \\ &+ (abc'' - ab''^2 + b'm^2 - a'b'c'')\,xy^2 \\ &+ (a'bc'' - bm^2 + 2b'b''m - a'b''^2 - b'^2c'')\,y^3 \\ &- 2ac''mx^2 + (2m^3 - ab''c'' - 2a'c''m)\,xy \\ &+ b''\,(m^2 - a'c'')\,y^2 \\ &- ac''^2x + c''\,(m^2 - a'c'')\,y = 0. \end{aligned}$$

The origin is evidently a point on the Hessian, but we will further show that it is a point of inflexion.

Turn the axes through an angle θ such that

$$\tan\theta = \frac{ac''}{m^2 - a'c''}.$$

We must have

$$X = c''\,(m^2 - a'c'')\,x - ac''^2y,$$
$$Y = ac''^2x + c''\,(m^2 - a'c'')\,y.$$

It will then be found that the coefficients of x and of x^2 vanish; i. e. the origin is a point of inflexion, and the new axis of x the tangent at it.

THE QUARTIC.

We write the general equation of the quartic in the form

$$\begin{aligned}
&ax^4 + by^4 + cz^4 \\
&+ 6fy^2z^2 + 6gz^2x^2 + 6hx^2y^2 \\
&+ 12lx^2yz + 12my^2zx + 12nz^2xy \\
&+ 4a'x^3y + 4a''x^3z + 4b'y^3x + 4b''y^3z + 4c'z^3x + 4c''z^3y = 0.
\end{aligned}$$

The equation of the Hessian is not given in any mathematical work in my possession. According to my calculation it is

$$\begin{aligned}
&(agh + 2a'a''l - al^2 - a''^2h - a'^2g)\, x^6 \\
&+ (bfh + 2b'b''m - bm^2 - b''^2h - b'^2f)\, y^6 \\
&+ (cfg + 2c'c''n - cn^2 - c''^2g - c'^2f)\, z^6 \\
&+ 2\,(ab'g + ahn + a'l^2 + 2a'a''m - 2alm - a'gh - a''^2b' - a'^2n)\, x^5y \\
&+ 2\,(a'bf + bhn + b'm^2 + 2b'b''l - 2blm - b'fh - a'b''^2 - b'^2n)\, xy^5 \\
&+ 2\,(bc''h + bfl + b''m^2 + 2b'b''n - 2bmn - b''fh - b'^2c'' - b''^2l)\, y^5z \\
&+ 2\,(b''cg + cfl + c''n^2 + 2c'c''m - 2cmn - c''fg - b''c'^2 - c''^2l)\, z^5y \\
&+ 2\,(a''cf + cgm + c'n^2 + 2c'c''l - 2cln - c'fg - a''c''^2 - c'^2m)\, z^5x \\
&+ 2\,(ac'h + agm + a''l^2 + 2a'a''n - 2aln - a''gh - a'^2c' - a''^2m)\, x^5z \\
&+ \left(\begin{array}{c} abg + afh - a''^2b - a'^2f + 2a'b'g + 2a'a''b'' + 2a'lm - 2ab''l \\ + 3hl^2 - 3gh^2 + 4ab'n - 4a'hn - 4am^2 + 6a''hm - 6a''b'l \end{array}\right) x^4y^2 \\
&+ \left(\begin{array}{c} abf + bgh - ab''^2 - b'^2g + 2a'b'f + 2a''b'b'' + 2b'lm - 2a''bm \\ + 3hm^2 - 3fh^2 + 4a'bn - 4b'hn - 4bl^2 + 6b''hl - 6a'b''m \end{array}\right) x^2y^4 \\
&+ \left(\begin{array}{c} bfg + bch - b'^2c - b''^2g + 2b''c''h + 2b'b''c' + 2b''mn - 2bc'm \\ + 3fm^2 - 3f^2h + 4bc''l - 4b''fl - 4bn^2 + 6b'fn - 6b'c''m \end{array}\right) y^4z^2 \\
&+ \left(\begin{array}{c} cfh + bcg - bc'^2 - c''^2h + 2b''c''g + 2b'c'c'' + 2c''mn - 2b'cn \\ + 3fn^2 - 3f^2g + 4b''cl - 4c''fl - 4cm^2 + 6c'fm - 6b''c'n \end{array}\right) y^2z^4 \\
&+ \left(\begin{array}{c} acf + cgh - ac''^2 - c'^2h + 2a''c'f + 2a'c'c'' + 2c'ln - 2a'cn \\ + 3gn^2 - 3fg^2 + 4a''cm - 4c'gm - 4cl^2 + 6c''gl - 6a''c''n \end{array}\right) z^4x^2 \\
&+ \left(\begin{array}{c} ach + afg - a'^2c - a''^2f + 2a''c'h + 2a'a''c'' + 2a''ln - 2ac''l \\ + 3gl^2 - 3g^2h + 4ac'm - 4a''gm - 4an^2 + 6a'gn - 6a'c'l \end{array}\right) x^4z^2 \\
&+ 2\left(\begin{array}{c} ab''g + ac''h - a''^2b'' - a'^2c'' + 2ab'c' + 2a'a''f \\ - 2a'c'h - 2amn - 2a''b'g - 2a'ln - 2a''lm \\ - 2afl + 3l^3 - 3ghl + 4a'gm + 4a''hn \end{array}\right) x^4yz
\end{aligned}$$

$$+\,2\left(\begin{array}{c} a''bf + bc'h - a''b''^{2} - b'^{2}c' + 2a'bc'' + 2b'b''g \\ - 2b'c''h - 2bln - 2a'b''f - 2b'mn - 2b''lm \\ - 2bgm + 3m^{3} - 3fhm + 4b'fl + 4b''hn \end{array}\right) y^{4}zx$$

$$+\,2\left(\begin{array}{c} a'cf + b'cg - a'c''^{2} - b'c'^{2} + 2c'c''h + 2a''b''c \\ - 2b''c'g - 2clm - 2a''c''f - 2c'mn - 2c''ln \\ - 2chn + 3n^{3} - 3fgn + 4c'fl + 4c''gm \end{array}\right) z^{4}xy$$

$$+\,2\left(\begin{array}{c} abn + ab'f + a'bg - a'fh - b'gh + 2a'b'n \\ + 2a''b''h - 2a'm^{2} - 2b'l^{2} - 2ab''m \\ - 2a''bl - 3h^{2}n + 6hlm \end{array}\right) x^{3}y^{3}$$

$$+\,2\left(\begin{array}{c} bcl + b''ch + bc''g - b''fg - c''fh + 2b'c'f + 2b''c''l \\ - 2b''n^{2} - 2c''m^{2} - 2bc'n - 2b'cm - 3f^{2}l + 6fmn \end{array}\right) y^{3}z^{3}$$

$$+\,2\left(\begin{array}{c} acm + ac'f + a''ch - c'gh - a''fg + 2a''c'm + 2a'c''g \\ - 2c'l^{2} - 2a''n^{2} - 2ac''n - 2a'cl - 3g^{2}m + 6gln \end{array}\right) z^{3}x^{3}$$

$$+\,2\left(\begin{array}{c} abc' - a''bg + 2ab'c'' + 2a''b'n + 2a'b'c' - 2a''m^{2} \\ - 2a'c''h - 2a'mn - 2a'fl + 3ghm - 3afm \\ - 3c'h^{2} + 4a'b''g - 4a''b''l + 5a''fh + 6l^{2}m - 6b'gl \end{array}\right) x^{3}y^{2}z$$

$$+\,2\left(\begin{array}{c} ab'c - a'ch + 2ab''c' + 2a'c'm + 2a''b'c' - 2a'n^{2} \\ - 2a''b''g - 2a''mn - 2a''fl + 3ghn - 3afn \\ - 3b'g^{2} + 4a''c''h - 4a'c''l + 5a'fg + 6l^{2}n - 6c'hl \end{array}\right) x^{3}yz^{2}$$

$$+\,2\left(\begin{array}{c} abc'' - ab''f + 2a'bc' + 2a'b''n + 2a'b'c'' - 2b''l^{2} \\ - 2b'c'h - 2b'ln - 2b'gm + 3fhl - 3bgl - 3c''h^{2} \\ + 4a''b'f - 4a''b''m + 5b''gh + 6lm^{2} - 6a'fm \end{array}\right) y^{3}x^{2}z$$

$$+\,2\left(\begin{array}{c} a'bc - b'ch + 2a''bc'' + 2b'c''l + 2a'b''c'' - 2b'n^{2} \\ - 2b''ln - 2a''b''f - 2b''gm + 3fhn - 3bgn - 3a'f^{2} \\ + 4b''c'h - 4b'c'm + 5b'fg + 6m^{2}n - 6c''hm \end{array}\right) y^{3}xz^{2}$$

$$+\,2\left(\begin{array}{c} ab''c - ac''f + 2a''b'c + 2a''c''m + 2a''b''c' - 2c''l^{2} \\ - 2c'hn - 2b'c'g - 2c'lm + 3fgl - 3chl - 3b''g^{2} \\ + 4a'c'f - 4a'c''n + 5c''gh + 6ln^{2} - 6a''fn \end{array}\right) z^{3}x^{2}y$$

$$+\,2\left(\begin{array}{c} a''bc - bc'g + 2a''b''c'' + 2b''c'l + 2a'b''c - 2c'm^{2} \\ - 2c''hm - 2a'c''f - 2c''lm + 3fgm - 3chm - 3a''f^{2} \\ + 4b'c''g - 4b'c'n + 5c'fh + 6mn^{2} - 6b''gn \end{array}\right) z^{3}y^{2}x$$

$$+\left\{\begin{array}{c} abc + 2ab''c'' + 2a''bc' + 2a'b'c - 3af^2 - 3ch^2 \\ - 3bg^2 - 6a''fm - 6b''gl - 6c''hl - 6a'fn \\ - 6b'gn - 6c'hm - 6a'c''m - 6b'c'l - 6a''b''n \\ + 10a'b''c' + 10a''b'c'' + 18fgh + 18lmn \end{array}\right\} x^2y^2z^2 = 0.$$

Let $$c = c' = c'' = 0.$$

The origin is a double point on the quartic, and the equation of the tangents at it is

$$gx^2 + 2nxy + fy^2 = 0.$$

The Hessian becomes

$$u_6 + u_5 + u_4 + u_3 + 3\,(n^2 - fg)\,(gx^2 + 2nxy + fy^2) = 0,$$

where u_n denotes the sum of the terms of the nth degree. The origin is also a double point on the Hessian unless $n^2 - fg = 0$.

The equation of the tangents to the Hessian at the double point is

$$gx^2 + 2nxy + fy^2 = 0,$$

and therefore the Hessian is acnodal or crunodal if the quartic is so. If $n^2 - fg = 0$, all terms below the third degree vanish, and the origin is a triple point on the Hessian. But this is the condition that the tangents at the double point on the quartic should be coincident; i. e. that the origin should be a cusp. The equation of the tangents at the triple point is

$$\begin{aligned} u_3 = (2gln - g^2m - a''fg)\,x^3 + (3fgl - b''g^2 - 2a''fn)\,x^2y \\ + (3fgm - a''f^2 - 2b''gn)\,xy^2 + (2fmn - f^2l - b''fg)\,y^2 = 0. \end{aligned}$$

If we write this equation in the form

$$(gx^2 + 2nxy + fy^2)\,[(2ln - gm - a''f)\,x + (2mn - fl - b''g)\,y] = 0,$$

we see that two tangents at the triple point coincide with the tangent at the cusp on the quartic.

Let $$c = c' = g = 0.$$

The origin is a point of inflexion on the quartic, and the case of the cubic suggests that it is also a point of inflexion on the Hessian. The Hessian is

$$\begin{aligned} u_6 + u_5 + u_4 + u_3 \\ - c''\,(ac'' + 6a''n)\,x^2 \\ + 2\,(3n^3 - a'c''^2 - 2a''c''f - 2c''ln)\,xy \\ + (2c''mn + 3fn^2 - c''^2h - 4c''fl)\,y^2 \\ - a''c''^2x + c''\,(n^2 - c''l)\,y = 0. \end{aligned}$$

Turn the axes through an angle θ such that

$$\tan\theta = \frac{a''c''}{n^2 - c''l}.$$

We must have

$$X = c''(n^2 - c''l)\,x - a''c''^2 y,$$
$$Y = a''c''^2 + c''(n^2 - c''l).$$

It will then be found that the coefficient of x vanishes, but that of x^2 does not. The Hessian passes through the point of inflexion, but does not seem to present any singularity at that point.

Let $$a'' = c = c' = g = 0.$$

The origin is now a point of undulation on the quartic, the tangent $y = 0$ meeting the curve in four consecutive points. The Hessian for this case is

$$\begin{aligned} &u_6 + u_5 + u_4 + u_3 \\ &- ac''^2x^2 + 2(3n^3 - a'c''^2 - 2c''ln)\,xy \\ &+ (2c''mn + 3fn^2 - c''^2h - 4c''fl)\,y^2 + c''(n^2 - c''l)\,y = 0. \end{aligned}$$

The tangent $y = 0$, at the point of undulation is also a tangent to the Hessian, but like the preceding case there is no singularity.

We will now consider singularities arising from the union of two or more simple singularities.

Let $$a'' = c = c' = c'' = g = n = 0.$$

The origin is, then, a tacnode on the quartic, arising from the union of two double points (Salmon, Art. 244). The Hessian for this case is

$$\begin{aligned} &- al^2x^6 + (bfh + 2b'b''m - b''^2h - bm^2 - b'^2f)\,y^6 \\ &+ 2(a'l^2 - 2alm)\,x^5y + 2(a'bf + b'm^2 + 2b'b''l - a'b''^2 - b'fh - 2blm)\,xy^5 \\ &+ (afh - a'^2f - 2ab''l + 3hl^2 - 4am^2 + 2a'lm)\,x^4y^2 \\ &+ \left\{\begin{matrix} abf - ab''^2 + 2a'b'f - 6a'b''m - 3fh^2 \\ + 3hm^2 - 4bl^2 + 2b'lm + 6b''hl \end{matrix}\right\} x^2y^4 \\ &+ 2(ab'f - a'fh - 2a'm^2 - 2b'l^2 - 2ab''m + 6hlm)\,x^3y^3 \\ &+ 2(bfl + b''m^2 - b''^2l - b''fh)\,y^5 \\ &- 2(2b''lm + 2a'b''f + 3fhm - 3m^3 - 4b'fl)\,y^4x \\ &- 2(2afl - 3l^3)\,x^4y \\ &- 2(ab''f + 2b''l^2 - 3fhl - 6lm^2 + 6a'fm)\,y^3x^2 \\ &- 2(2a'fl + 3afm - 6l^2m)\,x^3y^2 \\ &- (3f^2h - 3fm^2 + 4b''fl)\,y^4 \\ &- 6a'f^2y^3x - 3af^2x^2y^2 - 6f^2ly^3 = 0. \end{aligned}$$

To determine the form of the curve in the neighborhood of the origin we employ the method explained by Salmon (Art. 56).

We write $y = Ax^\beta$ and determine β by the condition, that it shall be positive, and that the indices of two or more terms shall be equal, and less than the index of any other term. Having found β, we determine A by equating to zero the quantity multiplying the terms with equal index.

In the present case we find $\beta = 2$; that is

$$y = Ax^2,$$

and the equation for the determination of A is

$$6f^2lA^3 + 3af^2A^2 + 2(2afl - 3l^3)A + al^2 = 0.$$

The roots of this equation are, in general, unequal, since the discriminant

$$12f^2l^2(864afl^6 + 56a^3f^3l^2 - 360a^2f^2l^4 - 432l^8 - 3a^4f^4)$$

does not vanish.

If a_1, a_2, a_3 are the roots of this equation, the curve near the origin resembles the curves $y = a_1x^2$, $y = a_2x^2$, $y = a_3x^2$; i.e. three branches of the Hessian touch each other at the origin, and it should be observed that, since A is determined by a cubic equation, at least one of these branches must be real.

If in the preceding equations a and l have the particular values

$$a = 6fp^2, \quad l = -fp;$$

the origin is a ramphoid cusp or node-cusp on the quartic arising from the union of a double point and a cusp (Salmon, Art. 244).

The equation for the determination of A becomes, after division by $-6f^3p$,

$$A^3 - 3pA^2 + 3p^2A - p^3 = (A - p)^3 = 0.$$

The cubic has three roots equal to p, and to the degree of approximation to which we have yet proceeded, the three branches of the Hessian coincide.

Let $$y = px^2 + Bx^\lambda.$$

Substituting this value of y in the equation of the Hessian, we can make the indices of two or more terms equal, and less than the index of any other term, by taking

$$\lambda = \tfrac{7}{3}.$$

The Hessian, then, near the origin resembles the curve

$$y = px^2 + Bx^{\frac{7}{3}},$$

and therefore consists of three osculating branches, two of which are imaginary.

If in addition to the above values

$$a = 6fp^2, \quad l = -fp;$$

we have $$a' = 3frp, \quad m = --fr;$$
the origin is an oscnode on the quartic arising from the coincidence of three double points as consecutive points of a curve of finite curvature.

And if we further have
$$\begin{aligned} b &= 6f(d^2 - M^2), \\ b' &= 3f(dr - L), \\ b'' &= -\,3fd; \end{aligned}$$
the origin is a tacnode-cusp on the quartic.

The cubic for the determination of A is the same for the three last cases, and therefore the ramphoid cusp, the oscnode, and the tacnode-cusp give rise to the same singularity on the Hessian.

Let $$c = c' = c'' = f = g = n = 0.$$

The quartic becomes
$$ax^4 + 4a'x^3y + 6hx^2y^2 + 4b'xy^3 + by^4 + 4a''x^3 + 12lx^2y + 12mxy^2 + 4b''y^3 = 0.$$
The origin is a triple point, the tangents at which are given by the equation
$$a''x^3 + 3lx^2y + 3mxy^2 + b''y^3 = 0.$$
The equation of the Hessian is
$$\begin{aligned} &u_6 + 2(a''l^2 - a''^2m)\,x^5 \\ &+ 2(3l^3 - 2a''lm - a''^2b'')\,x^4y \\ &+ 4(3l^2m - 2a''b''l - a''m^2)\,x^3y^2 \\ &+ 4(lm^2 - 2a''b''m - b''l^2)\,x^2y^3 \\ &+ 2(3m^3 - 2b''lm - a''b''^2)\,xy^4 \\ &+ 2(b''m^2 - b''^2l)\,y^5 = 0. \end{aligned}$$
The origin is then a quintuple point on the Hessian, the tangents at it being given by the equation
$$\begin{aligned} &a''(l^2 - a''m)\,x^5 \\ &+ (3l^3 - a''^2b'' - 2a''lm)\,x^4y \\ &+ 2(3l^2m - a''m^2 - 2a''b''l)\,x^3y^2 \\ &+ 2(3lm^2 - b''l^2 - 2a''b''m)\,x^2y^3 \\ &+ (3m^2 - a''b''^2 - 2b''lm)\,xy^4 \\ &+ b''(m^2 - b''l)\,y^5 = 0. \end{aligned}$$

This equation may be written
$$\begin{aligned} &(a''x^3 + 3lx^2y + 3mxy^2 + b''y^3) \\ &\quad \times [(l^2 - a''m)\,x^2 + (lm - a''b'')\,xy + (m^2 - b''l)\,y^2] = 0, \end{aligned}$$

which shows that three of the tangents at the quintuple point on the Hessian coincide with the tangents at the triple point on the quartic, the equation of the other two tangents being

$$(l^2 - a''m)\,x^2 + (lm - a''b'')\,xy + (m^2 - b''l)\,y^2 = 0.$$

Let $$a'' = c = c' = c'' = g = 0.$$

The origin is evidently a double point, the tangents being

$$\begin{gathered} y = 0, \\ fy + 2nx = 0. \end{gathered}$$

But it will be found that the tangent $y = 0$ meets the curve in four consecutive points, and is therefore a stationary tangent. The origin is, then, a flecnode resulting from the coincidence of a double point and a point of inflexion. The Hessian is

$$\begin{aligned} &u_6 + u_5 \\ &+ (3fm^2 - 3f^2h - 4b''fl - 4bn^2 + 2b''mn + 6b'fn)\,y^4 \\ &+ 2\,(3fhm - 2b'n^2 - 2b''ln - 3a'f^2 + 6m^2n)\,y^3x \\ &+ 3\,(6lmn - 2a'fn - af^2)\,y^2x^2 \\ &+ 2\,(6l^2n - 3afn - 2a'n^2)\,yx^3 - 4an^2x^4 \\ &+ 2\,(6fmn - 3f^2l - 2b''n^2)\,y^3 + 12mn^2y^2x + 12ln^2yx^2 \\ &+ 3fn^2y^2 + 6n^3xy = 0. \end{aligned}$$

The origin is a flecnode on the Hessian, and the tangents at it

$$\begin{gathered} y = 0, \\ fy + 2nx = 0 \end{gathered}$$

coincide with the tangents at the flecnode on the quartic.

Let $$a'' = b'' = c = c' = c'' = f = g = 0.$$

The axes $x = 0, y = 0$ are tangents at a double point, and since each meets the curve in four coincident points they are also stationary tangents. The origin is a bi-flecnode arising from the union of two points of inflexion. The Hessian for this case is

$$u_6 + u_5 + u_4 + 12ln^2x^2y + 12mn^2xy^2 + 6n^3xy = 0.$$

The origin is, then, a bi-flecnode on the Hessian, and the tangents at it coincide with the tangents at the bi-flecnode on the quartic.